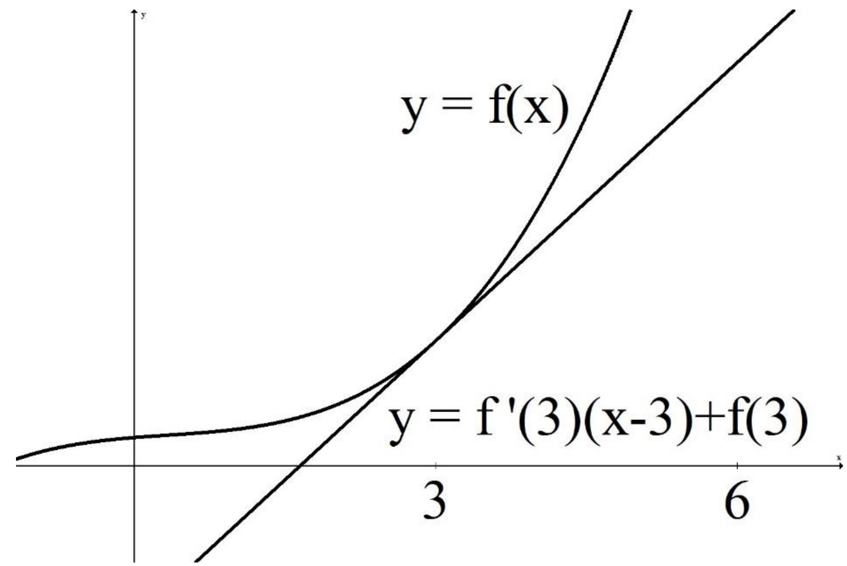
Closing Thu: 15.5
Closing Tue: Taylor Notes 1, 2, 3
Closing Next Thu: Taylor Notes 4, 5
Final is Saturday, March 12
5:00-7:50pm
KANE 130
Eight pages of questions, covers everything.

Taylor Notes 1 (TN 1): Tangent Line Error Bounds

Goal: Approximate functions with tangent lines and get error bounds. And begin a process of better and better approximations.



Def'n: We say the **first Taylor polynomial for f(x) based at b** (or the tangent line approximation) is

$$\mathbf{T_1(x) = f(b) + f'(b)(x-b)}$$

(This is just the tangent line to f(x) at x=b.)

Warm up: Before we discuss error bounds, let's talk about bounds and inequalities.
An upper **bound**, M, for a function is something that is always bigger than that function.

Examples: Find an upper **bound** for the functions below on the given intervals:

1. $|\sin(5x)|$ on $[0,2\pi]$

2. $|x-3|$ on $[1,5]$

3. $\left|\frac{1}{(2-x)^3}\right|$ on $[-1,1]$

4. $|\sin(x)+\cos(x)|$ on $[0,2\pi]$

5. $\left|\cos(2x) + e^{-2x} + \frac{6}{x}\right|$ on $[1,4]$

Tangent Linear Error Bound Theorem
(1st case of Taylor's Inequality)
If |f''(x)| ≤ M for all *x* values between *a* and *b*, then, for all *x* values between *a* and *b*, we have

ERROR = $|f(x) - T_1(x)| \leq \frac{M}{2}|x-b|^2$.

Note:

M = some upper bound on f''(x)

$|x-b|$ = the distance that x is away from b.

Proof sketch for x > b (for your own interest):
Start with $f(x) - f(b) = \int_b^x f'(t)dt$.

Do integration by parts in a clever way
(u = f'(t), dv = dt, du = f''(t), v = t – x) to get

$$f(x) - f(b) = f'(b)(x-b) - \int_b^x (t-x)f''(t)dt$$

Rearrange to get

$$f(x) - f(b) - f'(b)(x-b) = \int_b^x (x-t)f''(t)dt$$

so

ERROR: $|f(x) - T_1(x)| = \left|\int_b^x (x-t)f''(t)dt\right|$

Then note

$$\left|\int_b^x (x-t)f''(t)dt\right| \leq \int_b^x (x-t)|f''(t)|dt$$

$$\leq M\int_b^x (x-t)dt$$

$$= \frac{M}{2}(x-b)^2$$

To use the Tangent Line Error Bound:

1. Find f''(t).
2. Find upper bound for |f''(t)| on the interval. Call this M.
3. Use the theorem.
4. And plug in x = "an endpoint" to get a single number for a worst case upper bound.

Two types of error bound questions in the current homework:

A) Given interval, find error bound.
B) Given error bound, find interval.

Example:

Let f(x) = ln(x).

1. Find the 1st Taylor polynomial based at b=1.
2. Find a bound on the error over the interval

   J = [1/2, 3/2]
3. Find an interval around b = 1 where the error is less than 0.01.

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Entry Task: Let $f(x) = x^{1/3}$ and b = 8.

1. Find the 1st Taylor Polynomial for f(x) based at b.
2. Use Taylor's inequality to give a bound on the error over the interval J = [7,9].

Finishing Example From Last Time:

Let f(x) = ln(x) and b = 1.

Last time we found

1st Taylor Polynomial:

f(1) = 0, f'(x) = 1/x, f'(1) = 1, so

$T_1(x) = 0 + 1(x-1) = x-1$

Error Bound on J = [1/2,3/2]:

Step 1: $|f''(x)| = \left|-\frac{1}{x^2}\right| = \frac{1}{x^2} \leq M$ = ??

Step 2: Error $\leq \frac{M}{2}|x-1|^2 \leq$??

Now, find an interval about b = 1 where the error is less than 0.01.

(TN 2 and 3): Higher Order Approximations

The **2^{nd} Taylor Polynomial** (or quadratic approximation) is given by

$$T_2(x) = f(b) + f'(b)(x - b) + \frac{1}{2}f''(b)(x - b)^2$$

and the quadratic error bound states:

If $|f'''(x)| \leq M$ for all *x* values between *a* and *b*, then, for all *x* values between *a* and *b*, we have

$$\text{ERROR} = |f(x) - T_2(x)| \leq \frac{M}{6}|x - b|^3.$$

Example:

Find the second Taylor polynomial for $f(x) = x^{1/3}$ based at b = 8 AND find the error bound on the interval J = [7,9].

Taylor Approximation Idea:
If two functions have all the same derivative values, then they are the same function (up to a constant).

Let's compare derivatives of f(x) and $T_2(x)$.

$T_2(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2$
$T_2'(x) = f'(b) + \frac{1}{2}f''(b)2(x-b)$
$T_2''(x) = f''(b)$
$T_2'''(x) = 0$

Now plug in x = b to each of these, we see that, at x = b, f(x) and $T_2(x)$ have:

1. The same values: $f(b) = T_2(b)$.
2. Same 1st deriv: $f'(b) = T_2'(b)$.
3. Same 2nd deriv: $f''(b) = T_2''(b)$.

But after that they are no longer equal.

Questions:
Why did we need a ½ ?

What would $T_3(x)$ look like?

In general, **Taylor polynomial of degree n**:

$T_n(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2$
$+\frac{1}{3!}f'''(b)(x-b)^3 + \cdots + \frac{1}{n!}f^{(n)}(b)(x-b)^n$

We can write this in a cleaner way using signma notation:

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(b)(x-b)^k$$

And we have **Taylor's Inequality** (error bound):

If $|f^{(n+1)}(x)| \leq M$ for all *x* between *a* and *b*,
then, for all *x* values between *a* and *b*, we have
ERROR = $|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-b|^{n+1}$.

Note: For a fixed constant, *a*, the expression $\frac{a^k}{k!}$ goes to zero as k goes to infinity. So the expression $\frac{1}{(n+1)!}|x-b|^{n+1}$, will always go to zero as *n* gets bigger. Which means that the error goes to zero (unless something unusual is happening with M; we will see examples).